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ON CERTAIN GENERALIZED MODIFIED BETA OPERATORS

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ABSTRACT

In the present paper we study about the Stancu variants of modified Beta operators. We obtain some direct results in simultaneous approximation and asymptotic formula for these operators. We also modify these operators so as to preserve the linear moments, by applying the King's approach.

KEYWORDS: Simultaneous Approximation, Modified Beta Operators, King's Approach, Stancu Type Operators, Asymptotic Formula, Peetre's K-Functional

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1. INTRODUCTION

Stancu type generalization of Bernstein polynomials was introduced by Stancu [10] as

$$S_n^{\alpha}(f,x)=\sum_{k=0}^{n} f\binom{k}{n} p_{n,\alpha}^{k}(x),\ \ 0\le x\le 1 \tag{1.1}$$

where $p_{n,\alpha}^{k}(x)=\binom{n}{k}\dfrac{\prod_{s=0}^{k-1}(x+\alpha s)\prod_{s=0}^{n-k-1}(1-x+\alpha s)}{\prod_{s=0}^{n-1}(1+\alpha s)}$

As a special case if $\alpha\alpha=1$, we get the classical Bernstein polynomials. Starting with two parameters α and β satisfying $0\le\alpha\le\beta$ in the year 1983, Stancu [11] gave the other generalization of the operators (1.1) as follows

$$S_n^{\alpha,\beta}(f,x)=\sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{k+\beta}\right), \tag{1.2}$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$, (1.3)

is the Bernstein basis function.

In the year 2010, Buyukyazici and collaborators [1] and [2] have proposed the Stancu variants of several well known operators and estimated some direct results. Recently Gupta-Yadav [4] established some interesting results for BBS operators. Motivating with their work, we propose Stancu type generalization of modified Beta operators (1.1).

For $0 \leq \alpha \leq \beta$, Stancu type generalization for modified Beta operators is as follows

$$P_n^{\alpha,\beta}(f,x)=\left(\frac{n-1}{n}\right)\sum_{v=0}^{\infty}b_{n,v}(x)\int_0^{\infty}p_{n,v}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt,\quad x\in[0,\infty) \tag{1.4}$$

where

$$b_{n,v}(x)=\frac{1}{B(v+1,n)}x^v(1+x)^{-(n+v+1)},$$

$$p_{n,v}(t)=\binom{n+v-1}{v}t^v(1+t)^{-(n+v)}$$

The operators $P_n^{\alpha,\beta}$ are called modified Beta-Stancu operators. For $\alpha=\beta=0$, the operators (1.4) reduce to the modified Beta operators defined by Gupta-Ahmad [4], as

$$P_n(f,x)=\left(\frac{n-1}{n}\right)\sum_{v=0}^{\infty}b_{n,v}(x)\int_0^{\infty}p_{n,v}(t)f(t)dt,\quad x\in[0,\infty) \tag{1.5}$$

$b_{n,v}(x)$ and $p_{n,v}(t)$ are defined as above. Some approximation properties of these operators were discussed by Maheshwari [6] and [7], Maheshwari-Gupta [8], and Maheshwari-Sharma [9] etc.

In the analysis, Hewitt-Stromberg [5] showed that the operators P_n are linear positive operators. Also $P_n^{\alpha,\beta}(1,x)=1$, howsoever smooth the function may be, it turns out the order of approximation for the operators (1.4) are at best of $O(n^{-1})$.

2. BASIC RESULTS

In this section, we give certain lemmas, which will be useful for the proof of main theorem.

Lemma-1 [3]: If for $m\in Z^+$ and $x\in[0,\infty)$,

$$U_{n,m}(x)=\frac{1}{n}\sum_{v=0}^{\infty}b_{n,v}(x)\left(\frac{v}{n+1}-x\right)^m$$

then there exists the following recurrence relation-

$$(n+1)U_{n,m+1}(x)=x(1+x)\{U'_{n,m}(x)+mU_{n,m-1}(x)\}$$

Consequently

- $U_{n,m}(x)$ is a polynomial in x of degree $\leq m$.
- $U_{n,m}(x)=O\left(n^{-[m+1/2]}\right)$ where $[\xi]$ denotes the integral part of ξ.

Lemma-2 [3]: There exists the polynomial $q_{i,j,r}(x)$ independent of n and v, such that

$$x^r(1+x)^r\frac{d^r}{dx^r}\frac{x^v}{(1+x)^{n+v}}=\sum_{\substack{2i+j\leq r,\\ i,j\geq 0}}(n+1)^i[v-(n+1)x]^j q_{i,j,r}(x)\frac{x^v}{(1+x)^{n+v}}$$

Lemma-3: Let f be r times $(r = 1,2,3, \ldots)$ differentiable on R^{+}, then

$$\mathrm{P}^{(\mathrm{r})}_{\mathrm{n},\alpha,\beta}(f,x) = \frac{(n-r-1)!\,(n+r-1)!}{n!\,(n-2)!}\left(\frac{n}{n+\beta}\right)^{r}\sum_{v=0}^{\infty} b_{n+r,v}\,(x)\int_{0}^{\infty} p_{n-r,v+r}\,(t) f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

Proof: We have

$$\mathrm{P}^{(\mathrm{r})}_{\mathrm{n},\alpha,\beta}(f,x) = \left(\frac{n-1}{n}\right)\sum_{v=0}^{\infty} b^{(r)}_{n,v}\,(x)\int_{0}^{\infty} p_{n,v}\,(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt.$$

Using Leibnitz theorem for x

$$\mathrm{P}^{(\mathrm{r})}_{\mathrm{n},\alpha,\beta}(f,x) = \left(\frac{n-1}{n}\right)\sum_{i=0}^{r}\sum_{v=i}^{\infty}\binom{r}{i}\frac{(n+v+r-i)!}{(n-1)!\,(v-i)!}(-1)^{r-i}\frac{x^{v-i}}{(1+x)^{n+v+1+r-i}}$$

$$\times\int_{0}^{\infty} p_{n,v}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

$$= \left(\frac{n-1}{n}\right)\sum_{v=0}^{\infty}\frac{(n+v+r)!}{(n-1)!\,v!}.\frac{x^{v}}{(1+x)^{n+v+r+1}}\int_{0}^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}p_{n,v+i}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

$$= \frac{(n-1)(n+r-1)!}{\mathrm{n}!}\sum_{v=0}^{\infty} b_{n+r,v}\,(x)\int_{0}^{\infty}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}p_{n,v+i}(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

Again using Leibnitz theorem, we get

$$p^{(r)}_{n-r,v+r}(t) = \frac{(n-1)!}{(n-r-1)!}\sum_{i=0}^{r}(-1)^{i}\binom{r}{i}p_{n,v+i}(t)$$

Hence we get

$$\mathrm{P}^{(\mathrm{r})}_{,\alpha,\beta}(f,x) = \frac{(n-r-1)!\,(n+r-1)!}{(n-2)!\,n!}\sum_{v=0}^{\infty} b_{n+r,v}\,(x)\int_{0}^{\infty}(-1)^{r}p^{(r)}_{n-r,v+r}\,(t) f\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

Solving integrals by parts ‘r’ times, we have the required Lemma

$$P^{(r)}_{n,\alpha,\beta}(f,x) = \frac{(n-r-1)!\,(n+r-1)!}{n!\,(n-2)!}\left(\frac{n}{n+\beta}\right)^{r}\sum_{v=0}^{\infty} b_{n+r,v}\,(x)$$

$$\times\int_{0}^{\infty} p_{n-r,v+r}\,(t) f^{(r)}\left(\frac{nt+\alpha}{n+\beta}\right)dt$$

Lemma-4: Let Us Define

$$T^{\alpha,\beta}_{r,n,m}(x) = \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}\,(x)\int_{0}^{\infty} p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m}dt$$

Then, we have

$$T_{r,n,0}^{\alpha,\beta}(x) = 1, \tag{2.1}$$

$$T_{r,n,1}^{\alpha,\beta}(x) = \frac{(n-r-2)(\alpha-\beta x) + n\{(2r+3)x + (r+1)\}}{(n+\beta)(n-r-2)} \tag{2.2}$$

$$T_{n,r,2}^{\alpha,\beta}(x) = \frac{(\alpha-\beta x)^2(n-r-2) + 2n(\alpha-\beta x)\{(2r+3)x + (r+1)\}}{(n+\beta)^2(n-r-2)} + \left(\frac{n}{n+\beta}\right)^2 \times$$

$$\frac{x^2\begin{Bmatrix}(2n-1)+(2r+3)\\(2r+5)\end{Bmatrix} + x\begin{Bmatrix}(2n-1)+(2r+3)(r+2)\\+(2r+5)(r+1)\end{Bmatrix} + (r+1)(r+2)}{(n-r-2)(n-r-3)} \tag{2.3}$$

and the following recurrence relation is satisfied

$$\frac{n+\beta}{n}(n-r-m-2)T_{r,n,m+1}^{\alpha,\beta}(x)$$

$$= x(1+x)\left[T_{r,n,m}^{\alpha,\beta,'}(x) + \mathrm{m}T_{r,n,m-1}^{\alpha,\beta}(x)\right] + \left(\frac{\alpha}{n+\beta} - x\right)\left\{\left(\frac{\alpha}{n+\beta} - x\right)\frac{\mathrm{n}+\beta}{\mathrm{n}} - 1\right\}\mathrm{m}T_{r,n,m-1}^{\alpha,\beta}(x)$$

$$+ \left[(m+r+1) + \frac{\mathrm{n}+\beta}{\mathrm{n}}\left(\frac{\alpha}{n+\beta} - x\right)(\mathrm{n}-\mathrm{r}-2\mathrm{m}-2) + (n+r+1)x\right]T_{r,n,m}^{\alpha,\beta}(x) \tag{2.4}$$

Further, for all $x \in [0,\infty)$, we have

$$T_{r,n,m}^{\alpha,\beta}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$$

where $[\mu]$ is the integral part of μ.

Proof: We Have Some Identities to Use in Our Proof

- $x(1+x)b'_{n,v}(x) = [v-(n+1)x]b_{n,v}(x),$
- $\mathrm{t}(1+\mathrm{t})p'_{n,v}(\mathrm{t}) = [v-nt]p_{n,v}(\mathrm{t})$

Obviously (2.1)-(2.3) can be obtained by taking $m = 0,1,2$ respectively in $T_{r,n,m}^{\alpha,\beta}$. To prove recurrence formula, taking first derivative of $T_{r,n,m}^{\alpha,\beta}$ with respect to x,

$$T_{r,n,m}^{\alpha,\beta,'}(x) = \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b'_{n+r,v}(x)\int_0^{\infty} p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt - mT_{n,r,m-1}^{\alpha,\beta}(x)$$

$$\therefore\ T_{r,n,m}^{\alpha,\beta,'}(x) + mT_{n,r,m-1}^{\alpha,\beta}(x) = \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b'_{n+r,v}(x)\int_0^{\infty} p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta} - x\right)^m dt$$

Multiplying by $\mathrm{x}(1+\mathrm{x})$ on both sides and using identity (1), we have

$$x(1+x)\left[T_{n,r,m}^{\alpha,\beta,'}(x) + mT_{n,r,m-1}^{\alpha,\beta}(x)\right]$$

$$= \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} x(1+x)b'_{n+r,v}(x)\int_0^{\infty} p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$= \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}[v-(n+r+1)x]b_{n+r,v}(x)\int_0^{\infty} p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$= \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}(x)\int_0^{\infty}(v-(n+r+1)x)p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$= \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}(x)\times$$

$$\int_0^{\infty}[(v+r-(n-r)t)+(n-r)t-r-(n+r+1)x]p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$= \left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}(x)\int_0^{\infty} t(1+t)p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$+\frac{(n-r)(n-r-1)}{n+r}\sum_{v=0}^{\infty} b_{n+r,v}(x)\int_0^{\infty} tp_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$-[r+(n+r+1)x]T_{n,r,m}^{\alpha,\beta}$$

Substituting

$$t = \left(\frac{n+\beta}{n}\right)\left[\left(\frac{nt+\alpha}{n+\beta}-x\right)-\left(\frac{\alpha}{n+\beta}-x\right)\right]$$

and Using the Identity (2), We Have

$$x(1+x)\left[T_{n,r,m}^{\alpha,\beta,'}(x)+mT_{n,r,m-1}^{\alpha,\beta}(x)\right]$$

$$= \left(\frac{n+\beta}{n}\right)[\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}(x)\int_0^{\infty} p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt$$

$$-\left(\frac{\alpha}{n+\beta}-x\right).\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n-r,v}(x)\int_0^{\infty} p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt]$$

$$+\left(\frac{n+\beta}{n}\right)^2[\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}(x)\int_0^{\infty} p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+2} dt$$

$$-2\left(\frac{\alpha}{n+\beta}-x\right).\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty} b_{n+r,v}(x)\int_0^{\infty} p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt$$

$$+\left(\frac{\alpha}{n+\beta}-x\right)^2.\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n-r,v}(x)\int_0^{\infty}p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt]$$

$$+\left(\frac{n+\beta}{n}\right)(n-r)[\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt$$

$$-\left(\frac{\alpha}{n+\beta}-x\right).\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt]$$

$$-[r+(n+r+1)x]T_{n,r,m}^{\alpha,\beta}(x).$$

Now Solving the Integrals by Parts, We Get

$$x(1+x)\left[T_{n,r,m}^{\alpha,\beta,'}(x)+mT_{n,r,m-1}^{\alpha,\beta}(x)\right]$$

$$=-(m+1)\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$+m\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1}dt$$

$$+\left(\frac{n+\beta}{n}\right)[-(m+2)\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt$$

$$+2\left(\frac{\alpha}{n+\beta}-x\right)(m+1)\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt$$

$$-\left(\frac{\alpha}{n+\beta}-x\right)^2 m\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p'_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1}dt]$$

$$+\left(\frac{n+\beta}{n}\right)(n-r)[\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1}dt$$

$$-\left(\frac{\alpha}{n+\beta}-x\right).\left(\frac{n-r-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p_{n-r,v+r}(t)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt]$$

$$-[r+(n+r+1)x]T_{n,r,m}^{\alpha,\beta}(x).$$

$$=(m+1)\left[2\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)-1\right]T_{n,r,m}^{\alpha,\beta}(x)+\mathrm{m}\left(\frac{\alpha}{n+\beta}-x\right)\times$$

$$\left[1-\left(\frac{\alpha}{n+\beta}-x\right)\left(\frac{n+\beta}{n}\right)\right]T_{n,r,m-1}^{\alpha,\beta}(x)-(\mathrm{m}+2)\left(\frac{n+\beta}{n}\right)T_{n,r,m+1}^{\alpha,\beta}(x)$$

$$+\left(\frac{n+\beta}{n}\right)(n-r)\left[T_{n,r,m+1}^{\alpha,\beta}(x)-\left(\frac{\alpha}{n+\beta}-x\right)T_{n,r,m}^{\alpha,\beta}(x)\right]-[r+(n+r+1)x]T_{n,r,m}^{\alpha,\beta}(x).$$

On rearranging both sides, we get the required recurrence formula (2.4) for $m \geq 0$.

The Peetre's K-Function

The Peetre's K-functional is defined by

$$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\| : g \in W^2\}$$

where

$W^2 = \{g \in C_B[0,\infty): g', g'' \in C_B[0,\infty)\}$, $\exists$ a positive constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\delta), \delta > 0,$$

where the second order modulus of smoothness is given by

$$\omega_2(f,\delta^{1/2}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{0\leq x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Corollary 1: Let δ be a positive integer, then for all $n > \gamma > 0$, and $x \in [0,\infty)$, there exists a positive constant M depending upon m and x such that

$$\left(\frac{n-1}{n}\right)\sum_{v=0}^{\infty} b_{n,v}(x) \int_{|t-x|\geq\delta} p_{n,v}(t)t^{\gamma}dt \leq M\, n^{-m}, m \in N.$$

3. MAIN RESULTS

In this section we have some important theorems and obtain asymptotic formula.

Theorem 1: Let $f \in Q$ has a derivative of order $(r+2)$ at a fixed point $x \in (0,\infty)$. f is bounded on every finite subinterval of R^+ . For some $\gamma > 0$, $f(t) = O(t^{\gamma})$, as

$t \to \infty$, we have

$$\lim_{n\to\infty} n\left[P_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right] = r(2+r-\beta)f^{(r)}(x) + [(1+r+\alpha) + (3+2r-\beta)x]$$

$$\times f^{(r+1)}(x) + x(1+x)f^{(r+2)}(x).$$

Proof: Using Taylor Series Expansion, We Have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^{r+2}$$

where $\epsilon(t,x)(t-x)^{r+2}$ is of exponential order as $t \to \infty$ and $\epsilon(t,x) \to 0$ as $t \to x$. Now from Lemma 3, we have

$$n\left[P_{n,\alpha,\beta}^{(r)}(f,x) - f^{(r)}(x)\right]$$

$$= nP_{n,\alpha,\beta}^{(r)}\left(\sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i, x\right) + nP_{n,\alpha,\beta}^{(r)}(\epsilon(t,x)(t-x)^{r+2}, x) - nf^{(r)}(x)$$

$$= n\frac{(n-r-2)!\,(n+r)!}{n!\,(n-2)!}.\left(\frac{\mathrm{n}}{\mathrm{n}+\beta}\right)^{\mathrm{r}}$$

$$\times\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}.\left(\frac{n-\mathrm{r}-1}{n+r}\right)\sum_{v=0}^{\infty}b_{n+r,v}(x)\int_0^{\infty}p_{n-r,v+r}(t)\frac{d^r}{dx^r}\left(\frac{nt+\alpha}{n+\beta}-x\right)^i dt-f^{(r)}(x)\right]$$

$$+n\left[\frac{(n-r-2)!\,(n+r)!}{n!\,(n-2)!}.\left(\frac{n}{n+\beta}\right)^r-1\right]f^{(r)}(x)$$

$$n\left(\frac{n-1}{n}\right)\sum_{v=0}^{\infty}b_{n,v}^{(r)}(x)\int_0^{\infty}p_{n,v}(t)\epsilon\left(\frac{nt+\alpha}{n+\beta},x\right)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{r+2}dt$$

$$= n\left[\frac{(n-r-2)!\,(n+r)!}{n!\,(n-2)!}\left(\frac{n}{n+\beta}\right)^r-1\right]f^{(r)}(x)+\mathrm{n}T_{n,r,1}^{\alpha,\beta}(x)f^{(r+1)}(x)+\frac{\mathrm{n}}{2!}T_{n,r,2}^{\alpha,\beta}(x)$$

$$\times f^{(r+2)}(x)\ +E_{n,r}(x) \qquad (3.1)$$

where

$$E_{n,r}(x)=(\mathrm{n}-1)\sum_{v=0}^{\infty}b_{n,v}^{(r)}(x)\int_0^{\infty}p_{n,v}(t)\epsilon(\mathrm{t},x)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{r+2}dt$$

Tends to 0 as $n\to\infty$ to show this we suppose $I_n=x^r(1+x)^rE_{n,r}(x)$, so that $I_n\to 0$ as $n\to\infty$.

Hence using Lemma 2

$$|I_n|\le(\mathrm{n}-1)\sum_{v=0}^{\infty}\sum_{\substack{2i+j<r;\\ i,j\ge 0}}(n+1)^i\,|v-(n+1)x|^j q_{i,j,r}(x)b_{n,v}(x)$$

$$\times\int_0^{\infty}p_{n,v}(t)\epsilon(\mathrm{t},x)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{r+2}dt$$

$$\le(n-1)K(x)\sum_{\substack{2i+j<r;\\ i,j\ge 0}}(n+1)^i\sum_{v=0}^{\infty}b_{n,v}(x)\,|v-(n+1)x|^j\int_0^{\infty}p_{n,v}(t)|\epsilon(t,x)||t-x|^{r+2}dt$$

$$\le(n-1)K(x)\sum_{\substack{2i+j<r\\ i,j\ge 0}}(n+1)^i\left(\sum_{v=0}^{\infty}b_{n,v}(x)\,|v-(n+1)x|^{2j}\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{v=0}^{\infty}b_{n,v}(x)\left(\int_0^{\infty}p_{n,v}(t)|\epsilon(t,x)||t-x|^{r+2}dt\right)^2\right)^{\frac{1}{2}}$$

where

$$K(x)=\sup_{\substack{2i+j\le r;\\ i,j\ge 0}}\left|q_{i,j,r}(x)\right|$$

Here $q_{i,j,r}(x)$ represents certain polynomials in x and independent of n and v. For a given $\epsilon > 0$, there exists a $\delta(\epsilon, x) > 0$ such that $|\epsilon(t,x)| < \epsilon$ for $0 < |t-x| < \delta$; and for $|t-x| \geq \delta$, we observe that $|\epsilon(t,x)| \leq M|t-x|^{\gamma}$.

$$\therefore \left(\int_0^\infty p_{n,v}(t)|\epsilon(t,x)||t-x|^{r+2}dt\right)^2 \leq \left(\int_0^\infty p_{n,v}(t)dt\right)\left(\int_0^\infty p_{n,v}(t)\left(\epsilon(t,x)\right)^2(t-x)^{2r+4}dt\right)$$

$$\leq \left(\frac{1}{n-1}\right)\int_{|t-x|<\delta} p_{n,v}(t)\left(\epsilon(t,x)\right)^2(t-x)^{2r+4}dt$$

$$+\left(\frac{1}{n-1}\right)\int_{|t-x|\geq\delta} p_{n,v}(t)\left(\epsilon(t,x)\right)^2(t-x)^{2r+4}dt$$

From Corollary 1, we have

$$\sum_{v=0}^{\infty} b_{n,v}(x)\left(\int_0^\infty p_{n,v}(t)|\epsilon(t,x)||t-x|^{r+2}dt\right)^2$$

$$\leq \frac{1}{n-1}\sum_{v=0}^{\infty} b_{n,v}(x)\int_0^\infty p_{n,v}(t)\epsilon^2(t-x)^{2r+4}dt + \left(\frac{M^2}{n-1}\right)\sum_{v=0}^{\infty} b_{n,v}(x)\int_{|t-x|\geq\delta} p_{n,v}(t)(t-x)^{2r+2\gamma+4}dt$$

$$= \epsilon^2 O\left(n^{-(r+3)}\right) + O\left(n^{-(q+1)}\right),$$

Now using Lemma 1, taking $q > (r+2)$ and n to be large enough, we get

$$|I_n| \leq (n-1)\,K(x)\sum_{\substack{2i+j<r;\\ i,j\geq 0}} n^{i+j}\left[nO(n^{-j})\right]^{\frac{1}{2}}\left[\epsilon^2 O\left(n^{-(r+3)}\right) + O\left(n^{-(q+1)}\right)\right]^{\frac{1}{2}}$$

$$= \left[\epsilon^2 O(1) + O\left(n^{r+3-(q+1)}\right)\right]^{\frac{1}{2}}$$

$$\leq \epsilon O(1).$$

This shows that $I_n \to 0$ as $n \to \infty$.

Hence in order to prove the theorem, we substitute the values of $T_{n,r,1}^{\alpha,\beta}(x)$ and $T_{n,r,2}^{\alpha,\beta}(x)$ from Lemma 4 in (3.1).

Suppose that $C_B[0,\infty)$ be the space of real valued continuous bounded functions f on the interval $[0,\infty)$, and let the norm $\|.\|$ on the space $C_B[0,\infty)$ be defined as

$$\|f\| = \sup_{0\leq x<\infty} |f(x)|.$$

Also using K-functional, defined in Section 4 and for $f \in C_B[0,\infty)$ the usual modulus of continuity is given by

$$\omega(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{0\leq x<\infty} |f(x+h) - f(x)|$$

Therefore, let us modify the operators (1.5)as

$$\hat{P}_n^{\alpha,\beta}(f,x) = P_n^{\alpha,\beta}(f,x) - f\left(x + \frac{(n-2)(\alpha-\beta x) + n(1+3x)}{(n-2)(n+\beta)}\right) + f(x) \tag{3.2}$$

Theorem 2: Let $f \in C_B[0,\infty)$, then $\forall\, x \in [0,\infty)$, and $n \in N$, there exists an absolute constant $K > 0$ such that

$$\left|P_n^{\alpha,\beta}(f,x) - f(x)\right| \leq C\omega_2\big(f, \delta_n(x)\big) + \omega\left(f, \frac{(n-2)(\alpha-\beta x) + n(1+3x)}{(n-2)(n+\beta)}\right).$$

where

$$\delta_n{}^2(x) = \frac{(n-2)(\alpha-\beta x)^2 + 2n(\alpha-\beta x)(1+3x)}{(n+\beta)^2(n-2)} + \frac{2n^2[(n+7)x^2 + (n+5)x + 1]}{(n+\beta)^2(n-2)(n-3)}$$

$$+2\frac{(n-2)(\alpha-\beta x) + n(1+3x)}{n+\beta} - x^2$$

Proof: Let $g \in W^2$. Then from Taylor expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du, t\epsilon[0,\infty)\backslash$$

Therefore from Lemma 4, we have

$$\hat{P}_n^{\alpha,\beta}(g,x) - g(x) = g'(x)\hat{P}_n^{\alpha,\beta}(t-x,x) + \hat{P}_n^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du, x\right)$$

Hence

$$\left|\hat{P}_n^{\alpha,\beta}(g,x) - g(x)\right| \leq \left|P_n^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)\,du, x\right)\right|$$

$$\leq \int_x^{x+\frac{(n-2)(\alpha-\beta x)+n(1+3x)}{(n-2)(n+\beta)}} \left|x + \frac{(n-2)(\alpha-\beta x) + n(1+3x)}{(n-2)(n+\beta)} - u\right| |g''(u)|du$$

$$\leq \left[P_n^{\alpha,\beta}((t-x)^2, x)\right] + \left(x + \frac{(n-2)(\alpha-\beta x) + n(1+3x)}{(n-2)(n+\beta)}\right)^2 \|g''\|$$

$$= \delta_n{}^2(x)\|g''\|$$

For $\hat{P}_{n,\alpha,\beta}(f,x)$, we have $\hat{P}_n^{\alpha,\beta}(f,x) \leq 3\|f\|$.

$$\therefore \left|P_n^{\alpha,\beta}(f,x) - f(x)\right| \leq \left|P_n^{\alpha,\beta}(f-g,x) - (f-g)(x)\right| + \left|P_n^{\alpha,\beta}(g,x) - g(x)\right|$$

$$+\left|f\left(x + \frac{n(\alpha-\beta x)(1+3x)}{n+\beta}\right) - f(x)\right|$$

$$\leq 3\|f-g\| + \delta_n{}^2(x)\|g''\| + \left|f\left(x + \frac{(n-2)(\alpha-\beta x) + n(1+3x)}{(n-2)(n+\beta)}\right) - f(x)\right|$$

Taking infimum on RHS over all $g \in W^2$, we get

$$\left|P_n^{\alpha,\beta}(f,x)-f(x)\right| \leq 3K_2\left(f,\delta_n^2(x)\right)+\omega\left(f,\frac{(n-2)(\alpha-\beta x)+n(1+3x)}{(n-2)(n+\beta)}\right)$$

According to the property of Peetre's K-functional, we get

$$\left|P_n^{\alpha,\beta}(f,x)-f(x)\right| \leq C\omega_2\left(f,\delta_n(x)\right)+\omega\left(f,\frac{(n-2)(\alpha-\beta x)+n(1+3x)}{(n-2)(n+\beta)}\right)$$

which is the required result and hence completes the proof of theorem.

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